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CRITICAL POINT THEOREMS FOR INDEFINITE FUNCTIONALS.(U)
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Vieri Benci[1] and Paul H. Rabinowitz[2]

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ABSTRACT

A variational principle of a minimax nature is developed and used to prove the existence of critical points for certain variational problems which are indefinite. The proofs are carried out directly in an infinite dimensional Hilbert space. Special cases of these problems previously had been tractable only by an elaborate finite dimensional approximation procedure. The main applications given here are to Hamiltonian systems of ordinary differential equations where the existence of time periodic solutions is established for several classes of Hamiltonians.

AMS(MOS) Subject Classifications: 34C15, 34C25, 35L60, 47H15, 58E05, 58F05

Key words: variational problem, indefinite functional, critical point, Hamiltonian system, semilinear wave equation, linking , minimax, superquadratic, subquadratic.

Work Unit No. 1, Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Many functionals which arise in a variational formulation of physical problems are highly indefinite, i.e. are not bounded from above or below even modulo subspaces or submanifolds of finite dimension or codimension. E.g. in mechanics, this is often the case for a Lagrangian. In some recent research, finite dimensional approximation arguments have successfully handled some such questions. The present paper gives a direct infinite dimensional method for finding critical points of a class of such problems and applies it in particular to find periodic solutions of Hamiltonian's equations in mechanics under various hypotheses on the Hamiltonian.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

CRITICAL POINT THEOREMS FOR INDEFINITE FUNCTIONALS

Vieri Benci[1] and Paul H. Rabinowitz[2]

Introduction

In some recent papers, the existence of time periodic solutions has been established for a family of semilinear wave equations [1] and certain Hamiltonian systems of ordinary differential equations [2-4]. These equations have the common feature that formally they can be derived from variational problems. However these variational problems are indefinite in a very strong sense, in particular the corresponding functionals are not bounded from above or from below, even modulo subspaces or submanifolds of finite dimension or codimension. Due to the lack of an appropriate theory to obtain critical points directly, finite dimensional approximation arguments were employed in [1] - [4]. Critical point theorems in $\mathbb{R}^n$ provided approximate solutions to the equations and suitable estimates and comparison arguments allowed passage to a limit to solve the problem of interest.

The main purpose of this paper is to prove some abstract existence theorems for critical points of a real valued functional on a Hilbert space. These theorems provide a framework for a direct infinite dimensional treatment of some of the problems mentioned earlier. When applicable they permit a simpler treatment of these questions and clarify their underlying common structure. In more concrete situations the abstract existence theorems given here yield critical points which are weak solutions of the associated differential equations. Hence regularity becomes a separate question for us, unlike [1] - [4] where existence and regularity were obtained simultaneously in the course of the approximation procedure.

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Before continuing some notational preliminaries are necessary. Let E be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. The open ball of radius R about 0 in E will be denoted by B_R. The orthogonal complement of a subspace E_1 of E will be denoted by $E_1^{\perp}$.

Our main result is the following:

<u>Theorem 0.1:</u> Let E be a real Hilbert space, E_1 a closed subspace of E, and $E_2 = E_1^{\perp}$. Suppose that $f \in C^1(E, \mathbb{R})$ and satisfies

(f_1) $f(u) = \frac{1}{2}(Lu, u) + b(u)$ where $u = u_1 + u_2 \in E_1 \oplus E_2$, $Lu = L_1u_1 + L_2u_2$ and $L_i : E_i \to E_i$, $i = 1, 2$ is a (bounded) *linear self adjoint mapping*.

(f_2) b is weakly continuous and is uniformly differentiable on bounded subsets of E.

(f_3) If for a sequence (u_m), $f(u_m)$ is bounded from above and $f'(u_m) \to 0$ as $m \to \infty$, then (u_m) is bounded.

(f_4) There are constants $r_1, r_2, \rho, \alpha, \omega$ with $r_1 > \rho$, $\alpha > \omega$ and $r_1, r_2, \rho > 0$ and there is an $e \in \partial B_1 \cap E_1$ such that

(i) $f \geq \alpha$ on $S \equiv \partial B_\rho$

(ii) $f \leq \omega$ on ∂Q where $Q = \{re \mid 0 \leq r \leq r_1\} \oplus (B_{r_2} \cap E_2)$.

Then f possesses a critical value $c \geq \alpha$.

<u>Remarks 0.2:</u> In (f_2), b weakly continuous means whenever u_m converges weakly to u, $b(u_m) \to b(u)$. The uniform differentiability of

b on bounded sets means for any $R, \varepsilon > 0$, there is a $\delta = \delta(R, \varepsilon) > 0$ (and independent of u) such that if $u, u + v \in B_R$ and $\|v\| \le \delta$, then

$$|b(u + v) - b(u) - (b'(u), v)| \le \varepsilon \|v\|$$

where $b'(u)$ denotes the Frechet derivative of b at u. The form of f and (f_2) imply that f is also uniformly differentiable on bounded subsets of E. Using (f_2) and standard theorems [5], it follows that $B(u) \equiv b'(u)$ maps weakly convergent to strongly convergent sequences and a fortiori is compact. (In the usual fashion we are interpreting B as a mapping of E to E). Hypothesis (f_3) is a weakened version of the Palais-Smale condition (PS) which states that whenever $|f(u_m)|$ is bounded and $f'(u_m) \to 0$, then u_m possesses a convergent subsequence. In (f_4)(ii), ∂Q refers to the boundary of Q as a subset of span $\{e\} \oplus E_2$.

Theorem 0.1 is the third in a line of related abstract results. Suppose E is a real Banach space, (f_1) - (f_2) are omitted, and (f_3) is replaced by (PS). Then if $E_2 = \{0\}$, we essentially have a theorem of Ambrosetti and Rabinowitz [6], [7] while if E_2 is finite dimensional we have a variant of results of Rabinowitz [1], [8] and Benci [9]. Unlike these earlier papers, here both E_1 and E_2 are permitted to be infinite dimensional. This is the case of greatest interest to us and several examples will be given in §2-5. As in [6] - [9], a minimax characterization of the critical value c can be given and this will be done in §1.

The applications of the predecessors of Theorem 0.1 have been to problems involving functionals $b(u)$ which in an appropriate sense increase at a more rapid rate than quadratic. (Henceforth such problems will be called superquadratic). This behavior is hidden in our hypotheses in (f_4)(ii). When combined, the pair of hypotheses (f_4)(i) - (ii) imply a certain topological linking occurs between S and ∂Q. This linking implies $c \ge \alpha$ and in applications shows any critical point corresponding to c is nontrivial. These cryptic remarks will be made more precise later.

In §1, Theorem 0.1 will be obtained as a consequence of a more general result, Theorem 1.4, in which (f_4) is supplanted by an abstract linking condition. Stating matters thusly permits a simultaneous proof of Theorem 0.1 and another result, Theorem 1.33 suitable for functionals with subquadratic nonlinearities. Here subquadratic refers to less rapid growth than quadratic in an appropriate sense. A simpler version of the latter type of result was obtained in [10]. (See also [11] - [13]).

A model problem which gives a simple illustration of Theorem 0.1 for an elliptic system of partial differential equations is given in §2. Then the abstract theorems are applied to obtain existence results for superquadratic (§3) and subquadratic (§4) Hamiltonian systems of ordinary differential equations. Lastly in §5 we indicate how to apply the theory to semilinear hyperbolic partial differential equations.

We thank Michael Crandall for helpful conversations.

§1. The abstract theorems.

This section is devoted to the proof of a general critical point theorem and specializations of the result appropriate for sub- and superquadratic problems. We begin with a notion of linking.

Mappings $h:[0,1]\times E \to E$ will be denoted by $h(t,u)$ or $h_t(u)$. In the sequel E always denotes a real Hilbert space, $E = E_1 \oplus E_2$, and if $u \in E$, $u = u_1 + u_2$ with $u_i \in E_i$, $i = 1,2$. Further set $P_i u \equiv u_i$, $i = 1,2$.

Let Σ denote the class of mappings $\phi \in C([0,1]\times E, E)$ for which $P_2\phi_t(u) = u_2 - W_t(u)$ with W_t compact for $t \in [0,1]$ and $\phi_0(u) = u$. Let S and Q be Hilbert manifolds, Q having a boundary, ∂Q. We say S and ∂Q <u>link</u> if whenever $\phi \in \Sigma$ and $\phi_t(\partial Q)\cap S = \emptyset$ for all $t \in [0,1]$, then $\phi_t(Q)\cap S \neq \emptyset$ for all $t\in[0,1]$. A somewhat inaccurate but geometrically useful picture to keep in mind is of S and ∂Q linking if every Hilbert manifold modelled on Q and sharing the same boundary intersects S. The following two lemmas provide examples of linking sets.

<u>Lemma 1.1:</u> Suppose $Q = B_R \cap E_2$, $q \in Q$, and $S = q + E_1$. Then S and ∂Q link.

<u>Proof:</u> Let $\phi \in \Sigma$ be such that $\phi_t(\partial Q)\cap S = \emptyset$ for all $t \in [0,1]$. We claim $\phi_t(Q)\cap S \neq \emptyset$ or equivalently there is a $q_t \in Q$ such that $P_2\phi_t(q_t) = q$ for all $t\in[0,1]$. Observe that for $u \in E_2$, $\Psi_t(u) \equiv P_2\phi_t(u) = u - W_t(u)$ has the appropriate form to apply the theory of degree of Leray-Schauder. We denote the Leray-Schauder degree of a map Ψ with respect to a bounded open set Ω and a point $a \notin \Psi(\partial\Omega)$ by $d(\Psi,\Omega,a)$. Thus consider $d(\Psi_t,Q,q)$. Since $\phi_t(\partial Q)\cap S = \emptyset$, this degree is well defined for all $t\in[0,1]$. By the homotopy invariance property of degree, $d(\Psi_t,Q,q) = d(\Psi_0,Q,q)$. Since $\Psi_0(u) = u$ and $q \in Q$, $d(\Psi_0,Q,q) = 1$. Hence there exists a point q_t as desired for each $t\in[0,1]$.

<u>Remark 1.2:</u> The linking obtained in Lemma 1.1 is of the type which often occurs in problems having subquadratic nonlinearities as will be seen from the applications given in §4. The next lemma describes a more subtle type of linking arising in superquadratic problems. Examples will be given in §2-3.

<u>Lemma 1.3:</u> Let $e \in \partial B_1 \cap E_1$ and $r_1 > \rho > 0$. If $S = \partial B_\rho \cap E_1$ and $Q = \{re \mid r \in [0,r_1]\} \oplus (B_{r_2}\cap E_2)$, then S and ∂Q link.

Proof: Let $\phi\in\Sigma$ and suppose that $\phi_t(\partial Q)\cap S = \emptyset$ for all $t\in[0,1]$. We must show $\phi_t(Q)\cap S\neq\emptyset$ or equivalently $P_2\phi_t(q_t) = 0$ and $\|P_1\phi_t(q_t)\| = \|\phi_t(q_t)\| = \rho$ for some $q_t \in Q$ and each $t\in[0,1]$. Let $u \in E_2$, $r \in \mathbb{R}$, and set

$$\Psi_t(r,u) = (\|P_1\phi_t(u+re)\| - \rho,\ P_2\phi_t(u+re)) .$$

Then as in Lemma 1.1, $d(\Psi_t,Q,0)$ is well defined for $t\in[0,1]$ and $d(\Psi_t,Q,0) = d(\Psi_0,Q,0) = 1$ since $\Psi_0(r,u) = (r-\rho,u)$. Hence the result follows.

A general critical point theorem can now be stated.

<u>Theorem 1.4:</u> Let E be a real Hilbert space, E_1 a closed subspace of E, and $E_2 = E_1^{\perp}$. Let $f \in C^1(E, \mathbb{R})$ and satisfy (f_1) - (f_3) and

$(\bar{f}_4)$ There exist Hilbert manifolds $S, Q \subset E$, Q having a boundary, constants $\alpha > \omega$, and $v \in E_2$ such that

(i) $S \subset v + E_1$ and $f \geq \alpha$ on S,

(ii) $f \leq \omega$ on ∂Q,

(iii) S and ∂Q link.

Then f possesses a critical value $c \geq \alpha$.

The proof of Theorem 1.4 will be accomplished in several steps. Roughly speaking, we obtain c as a minimax of f over a class of surfaces modelled on Q. To be a bit more precise, using (f_1) - (f_2), we obtain Theorem 1.5, a variant of a standard deformation result. The properties of the mappings given by Theorem 1.5 suggest how to construct the appropriate class of surfaces and define c. Applying $(\bar{f}_4)$, we show $c \geq \alpha$. Lastly Theorem 1.5 and (f_3) or more properly a local version of (f_3) relative to c are employed to show c is indeed a critical value of f.

To begin this program, let Γ denote the set of mappings $h \in C([0,1] \times E, E)$ such that:

(Γ_1) $h_t(u) = U_t(u) + K_t(u)$ where $U, K \in C([0,1] \times E, E)$, U_t is a homeomorphism of E onto E and K_t is compact for each $t \in [0,1]$,

(Γ_2) $U_0(u) = u$, $K_0(u) = 0$,

(Γ_3) $P_i U_t(u) = U_t(P_i(u))$, $i = 1, 2$

(Γ_3) h_t maps bounded sets to bounded sets.

For $h \in \Gamma$, let $h_t^j(u)$ denote the j-fold composite of h with itself, i.e. $h_t^1(u) = h_t(u)$, $h_t^2(u) = h_t(h_t(u))$, and $h_t^j(u) = h_t(h_t^{j-1}(u))$ for $j > 1$. Let $\mathcal{B}_\tau = (B_\tau \cap E_1) \oplus (B_\tau \cap E_2)$.

<u>Theorem 1.5:</u> Let $f \in C^1(E, \mathbb{R})$ and satisfy (f_1) - (f_2). Then for any R, $\gamma > 0$ and $\varepsilon \in (0, \frac{1}{4})$, there is a $k \in \mathbb{N}$ and $\eta \in \Gamma$ such that

1° $f(\eta_t^k(u)) \leq f(u) + \gamma$ for all $u \in \mathcal{B}_{R+2}$ and $t \in [0,1]$.

2° If $c \in \mathbb{R}$ and $\|f'(w)\| \geq \sqrt{2\varepsilon}$ for all $w \in \mathcal{B}_{R+1} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$ then $f(\eta_1^k(u)) \leq c - \frac{\varepsilon}{2}$ whenever $u \in \mathcal{B}_R \cap f^{-1}([c-\varepsilon, c+\varepsilon])$.

Proof: Theorems like this result form a basic part of any minimax proof of the existence of critical points of a real valued functional. See e.g. [7]. The mapping η is usually determined by solving an appropriate differential equation involving $-f'(\eta)$. Such an approach seems to fail here since it does not give an η satisfying (Γ_1) - (Γ_3) which are crucial for our later purposes.

Choose $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ if $s \leq R+1$, $\chi(s) = 0$ if $s \geq R+2$, and $\chi'(s) < 0$ if $s \in (R+1, R+2)$. We can further assume $\chi(s) \leq (R+2-s)^2$ for $s \in [R+\frac{3}{2}, R+2]$. For $u = u_1 + u_2 \in E_1 \oplus E_2$, set $V_i(u) = \chi(\|u_i\|) P_i f'(u)$, $i = 1, 2$

and $V(u) = V_1(u) + V_2(u)$. Note that $V(u) = f'(u)$ in $\mathcal{B}_{R+1}$ and by (f_1) - (f_2), there is a constant $M = M(R)$ such that $\|f'(u)\| \le M$ for $u \in \mathcal{B}_{R+2}$. Set

$$\bar{\varepsilon} = M^{-1} \min(\gamma, \tfrac{\varepsilon}{2}) . \tag{1.6}$$

Since f is uniformly differentiable, there is a $\delta = \delta(\bar{\varepsilon}, R) > 0$ such that

$$|f(u+v) - f(u) - (f'(u), v)| \le \bar{\varepsilon}\|v\| \tag{1.7}$$

if $u, u+v \in \mathcal{B}_{R+2}$ and $\|v\| \le \delta$. We can assume $\delta \le 1$. Choose $k \in \mathbb{N}$ such that

$$\frac{1}{k} < \min\left(\frac{\delta}{2M}, \frac{1}{8(R+2)(1+\max_R |\chi'(s)|)(\|L_1\| + \|L_2\|)}\right) \tag{1.8}$$

Now define $\eta_t(u) = u - \frac{t}{k} V(u)$. We claim $\eta \in \Gamma$. Indeed observe that

$$P_1 \eta_t(u) = (I - \tfrac{t}{k}\chi(\|u_1\|)L_1)u_1 - \tfrac{t}{k}\chi(\|u_1\|)P_1 B(u)$$

where I denotes the identity map in E. Hence setting

$$U_t(u) = \sum_{i=1}^{2} (I - \tfrac{t}{k}\chi(\|u_i\|)L_i)u_i$$

and

$$K_t(u) = -\frac{t}{k}\sum_{i=1}^{2} \chi(\|u_i\|)\, P_i B(u),$$

it is clear that K_t is compact and η satisfies (Γ_2) and (Γ_3). The form of η and (f_1) - (f_2) imply that (Γ_4) is satisfied. Lastly to complete the verification of (Γ_1), we must show U_t in a homeomorphism

of E onto E. By (Γ_3), it suffices to show that $P_i U_t$ is a homeomorphism of E_i onto E_i, $i = 1, 2$. Thus let $u, v \in E_i$. For any $t \in [0,1]$,

$$\frac{t}{k}\|\chi(\|u\|)L_i u - \chi(\|v\|)L_i v\| \le \frac{1}{2}\|u - v\| \tag{1.9}$$

via (1.8) and some simple estimates. Hence given any $w \in E_i$, $\mathcal{L}_w(u) \equiv \frac{t}{k}\chi(\|u\|)L_i u$ is a contraction on E_i. It then follows from the contracting mapping theorem that $\mathcal{L}_w(u)$ has a unique fixed point u. This observation and (1.8) easily imply $P_i U_t$ is a homeomorphism on E_i and $\eta \in \Gamma$.

Next observe that $\eta_t : \mathcal{B}_{R+2} \to \mathcal{B}_{R+2}$. In fact if $u = u_1 + u_2 \in \mathcal{B}_{R+2}$,

$$\|P_i \eta_t(u) - u_i\| = \frac{t}{k}\|V_i(u)\| \le \frac{M}{k}\chi(\|u_i\|) . \tag{1.10}$$

Since $\text{dist}(u, \partial\mathcal{B}_{R+2})$, the distance from u to $\partial\mathcal{B}_{R+2}$ satisfies

$$\text{dist}(u, \partial\mathcal{B}_{R+2}) = \min_{i=1,2} R + 2 - \|u_i\| ,$$

if we show the right hand side of (1.10) does not exceed $R + 2 - \|u_i\|$, $i = 1, 2$, $\mathcal{B}_{R+2}$ is an invariant set for η_t. If $u_i \in B_{R+\frac{3}{2}}$,

$$Mk^{-1}\chi(\|u_i\|) \le \frac{1}{2} \le R + 2 - \|u_i\|$$

via (1.8) while if $\|u_i\| \ge R + \frac{3}{2}$,

$$R + 2 - \|u_i\| \ge (R + 2 - \|u_i\|)^2 \ge Mk^{-1}\chi(\|u_i\|)$$

by the choice of χ and M.

Now 1° of Theorem 1.5 can be verified. By (1.8) again, $\|\eta_t(u) - u\| \leq \delta$ for all $u \in \mathcal{B}_{R+2}$. Hence by (1.7)

$$f(\eta_t(u)) \leq f(u) - \frac{t}{k}(f'(u), V(u)) + \frac{\bar{\varepsilon} t}{k}\|V(u)\| . \tag{1.11}$$

Since $E_2 = E_1^{\perp}$,

$$(f'(u), V(u)) = \chi(\|u_1\|)\|P_1 f'(u)\|^2 + \chi(\|u_2\|)\|P_2 f'(u)\|^2 \geq 0 . \tag{1.12}$$

Thus by (1.6) and (1.11),

$$f(\eta_t(u)) \leq f(u) + \frac{\gamma}{k} . \tag{1.13}$$

Since $\mathcal{B}_{R+2}$ is invariant under η_t, 1° obtains by iterating (1.13) k times.

Finally to prove 2°, three cases are considered:

Case i: $\eta_1^j(u) \in \mathcal{B}_{R+1} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$ for $1 \leq j \leq k$.

Since $V(u) = f'(u)$ in $\mathcal{B}_{R+1}$, by (1.11) and (1.6):

$$\begin{aligned} f(\eta_1^k(u)) - f(u) &= \sum_{j=1}^{k} f(\eta_1^j(u)) - f(\eta_1^{j-1}(u)) \\ &\leq \sum_{j=1}^{k} [-\frac{1}{k}\|f'(\eta_1^{j-1}(u))\|^2 + \frac{\varepsilon}{2k}] \leq -2\varepsilon + \frac{\varepsilon}{2} = -\frac{3\varepsilon}{2} \end{aligned} \tag{1.14}$$

which implies 2°.

Case ii: $\eta_1^j(u) \in \mathcal{B}_{R+1} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$ for $1 \leq j \leq m-1$ but $\eta_1^m(u) \notin f^{-1}([c-\varepsilon, c+\varepsilon])$.

Estimating as in (1.14) shows that $f(\eta_1^j(u)) < f(\eta_1^{j-1}(u))$ for $1 \leq j \leq m$. Thus $\eta_1^m(u) \notin f^{-1}([c-\varepsilon, c+\varepsilon])$ implies that

$f(\eta_1^m(u)) < c - \varepsilon$. Therefore

$$\begin{aligned} f(\eta_1^k(u)) &= f(\eta_1^m(u)) + \sum_{j=m+1}^{k} (f(\eta_1^j(u)) - f(\eta_1^{j-1}(u))) \\ &\leq c - \varepsilon + \frac{k-m}{k}\,\frac{\varepsilon}{2} \leq c - \frac{\varepsilon}{2} \end{aligned}$$

via (1.11), (1.12), and (1.6).

Case iii: $\eta_1^j(u) \in \mathcal{B}_{R+1} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$ for $1 \leq j \leq m-1$ but $\eta_1^m(u) \notin \mathcal{B}_{R+1}$.

Then

$$\begin{aligned} 1 &\leq \|\eta_1^m(u)\| - \|u\| \leq \|\eta_1^m(u) - u\| \leq \sum_{j=1}^{m} \|\eta_1^j(u) - \eta_1^{j-1}(u)\| \\ &= \frac{1}{k}\sum_{j=1}^{m} \|f'(\eta_1^{j-1}(u))\| \leq \frac{m^{\frac{1}{2}}}{k}\left(\sum_{j=1}^{m} \|f'(\eta_1^{j-1}(u))\|^2\right)^{\frac{1}{2}} . \end{aligned} \tag{1.15}$$

Employing (1.11) and (1.6) in (1.15) yields

$$\begin{aligned} 1 &\leq \frac{m^{\frac{1}{2}}}{k}\left[\sum_{j=1}^{m} k(f(\eta_1^{j-1}(u)) - f(\eta_1^j(u))) + \frac{\varepsilon}{2}\right]^{\frac{1}{2}} \\ &\leq \frac{m^{\frac{1}{2}}}{k}\left[k(f(u) - f(\eta_1^m(u)) + \frac{m\varepsilon}{2}\right]^{\frac{1}{2}} . \end{aligned}$$

Therefore since $f(\eta_1^m(u)) < f(u)$ as in Case ii,

$$\begin{aligned} 1 &\leq \frac{m}{k}(f(u) - f(\eta_1^m(u)) + \frac{m^2}{k^2}\,\frac{\varepsilon}{2} \leq f(u) - f(\eta_1^m(u)) + \frac{\varepsilon}{2} \\ &\leq c + \varepsilon + \frac{\varepsilon}{2} - f(\eta_1^m(u)) \end{aligned}$$

or

$$f(\eta_1^m(u)) \leq c + \frac{3\varepsilon}{2} - 1 \leq c + \frac{3\varepsilon}{2} - 4\varepsilon < c - \varepsilon .$$

Hence $f(\eta_1^k(u)) \le c - \frac{\varepsilon}{2}$ by Case ii. The proof is complete.

Motivated by the properties of η, we introduce a class of maps, Λ, that will be used to determine c. Let $G_s = \{u \in E \mid f(u) \le s\}$. Let Λ denote the set of $h \in C([0,1] \times E, E)$ for which there is a $\beta \in (0, \frac{\alpha - \omega}{2})$, $m \in \mathbb{N}$, and $h^{(1)}, \ldots, h^{(m)} \in \Gamma$ such that $h = h^{(1)} \circ \ldots \circ h^{(m)}$ and $h_t(\partial Q) \subset G_{\frac{\alpha+\omega}{2} - \beta}$. Observe that $\Lambda \ne \emptyset$ since $h(u) = u \in \Lambda$ by $(\overline{\Gamma}_4)$ (ii).

Now define

(1.16) $$c = \inf_{h \in \Lambda} \sup_{u \in \overline{Q}} f(h_1(u)) .$$

We will show c is a critical value of f. As a preliminary result we require:

<u>Proposition 1.17</u>: If f satisfies $(\overline{\Gamma}_4)$, then $c \ge \alpha$.

Proof: It suffices to show that

(1.18) $$h_1(\overline{Q}) \cap S \ne \emptyset$$

for all $h \in \Lambda$. For then if $y \in h_1(\overline{Q}) \cap S$,

(1.19) $$\sup_{u \in \overline{Q}} f(h_1(u)) \ge f(y) \ge \inf_{w \in S} f(w) \ge \alpha$$

via $(\overline{\Gamma}_4)$(i). Since (1.19) holds for all $h \in \Lambda$, (1.16) and (1.19) imply $c \ge \alpha$.

The proof of (1.18) follows from the stronger assertion that

(1.20) $$h_t(\overline{Q}) \cap S \ne \emptyset$$

for all $h \in \Lambda$ and $t \in [0,1]$. Since $S - v \subset E_1$ by $(\overline{\Gamma}_4)$(i), (1.20) is equivalent to finding, for each $t \in [0,1]$, a $u \in \overline{Q}$ such that

(1.21) $$\begin{cases} P_1 h_t(u) \in S - v \\ P_2 h_t(u) = v \end{cases} .$$

We will solve (1.21) by converting it to an equivalent problem to which the linking hypotheses can be applied. Suppose first that $h \in \Lambda$ with the corresponding $m = 1$. Letting $u = u_1 + u_2 \in E_1 \oplus E_2$ as usual, by (Γ_1) and (Γ_3), (1.21) becomes

(1.22) $$\begin{cases} \text{(i)} \quad P_1 h_t(u) \in S - v \\ \text{(ii)} \quad P_2 h_t(u) = U_t(u_2) + P_2 K_t(u) = v \end{cases} .$$

More generally, suppose (1.22) (ii) is replaced by

(1.23) $$P_2 h_t(u) = P_2 Z_t(u)$$

where $Z_t(u)$ is compact and $Z_0(u) = v$. By (Γ_1) and (Γ_3) (1.23) is equivalent to

(1.24) $$u_2 = U_t^{-1}(-P_2(K_t(u) - Z_t(u))) \equiv P_2 Y_t(u)$$

where Y_t is compact and $Y_0(u) = v$. Suppose we have shown by induction that whenever $h \in \Lambda$ with $m = n - 1$, (1.23) is equivalent to

(1.25) $$u_2 = P_2 Y_t(u)$$

with Y_t compact and $Y_0(u) = v$. Let $h \in \Lambda$ with $m = n$ so $h = h^{(1)} \circ \ldots \circ h^{(n)}$. Let $\hat{h} = h^{(2)} \circ \ldots \circ h^{(n)}$. Hence

$h = h^{(1)} \circ \hat{h}$ and as above, the equation $P_2 h_t(u) = v$ is equivalent to

(1.26) $$P_2 \hat{h}_t(u) = (U_t^{(1)})^{-1} \, (-P_2 K_t^{(1)} (\hat{h}_t(u)) + v)$$

where $h^{(1)} = U^{(1)} + K^{(1)}$ and the right hand side of (1.26) is compact and equals v when $t = 0$. Hence by induction hypothesis there is a compact Y such that (1.26) is equivalent to solving (1.25).

Now set $\Phi_t(u) = P_1 h_t(u) + u_2 - P_2 Y_t(u) + v$. It is evident that $\Phi \in \Sigma$, $P_1 \Phi_t = P_1 h_t$ and by our above remarks, $P_2 \Phi_t = v$ is equivalent to $P_2 h_t = v$. Therefore $\Phi_t(u) \in S$ if and only if $h_t(u) \in S$ and to complete the proof we need only show

(1.27) $$\Phi_t(Q) \cap S \neq \emptyset$$

for all $t \in [0,1]$. Since $\Phi \in \Sigma$ and S and ∂Q link by $(\bar{I}_4)$ (iii), (1.27) follows if

(1.28) $$\Phi_t(\partial Q) \cap S = \emptyset .$$

Suppose to the contrary that there is a $u \in \partial Q$ and $t \in [0,1]$ such that $\Phi_t(u) \in S$. Then $h_t(u) \in S$. But $h_t(\partial Q) \subset G_{\frac{\alpha+\omega}{2}-\beta}$ since $h \in \Lambda$ while $S \cap G_{\frac{\alpha+\omega}{2}-\beta} = \emptyset$ by $(\bar{I}_4)$ (i), a contradiction. Thus (1.28) is satisfied and the proof is complete.

Now we can prove that c is a critical value of f.

<u>Theorem 1.29</u>: Let E be a real Hilbert space, E_1 a closed subspace of E, $E_2 = E_1^{\perp}$, and $f \in C^1(E, \mathbb{R})$ with f satisfying (f_1), (f_2), and $(\bar{f}_4)$. If c is defined by (1.16) and

(f_5) there is an $a > 0$ such that $(u_m) \subset f^{-1}([c-a, c+a])$ and $f'(u_m) \to 0$ implies that (u_m) is bounded

also holds, then c is a critical value of f.

Proof: Observe first that since $h(u) = u \in \Lambda$, c is finite via (f_2). Moreover $c \geq \alpha$ by Proposition 1.17. If c is not a critical value of f, $f'(u) \neq 0$ for all $u \in f^{-1}(c)$. We claim there is an $\varepsilon > 0$ such that $\|f'(u)\| \geq \sqrt{2\varepsilon}$ for all $u \in f^{-1}([c-\varepsilon, c+\varepsilon])$. If not, there is a sequence of positive $\varepsilon_n \to 0$ and $u_n \in f^{-1}([c-\varepsilon_n, c+\varepsilon_n])$ such that $\|f'(u_n)\| < \sqrt{2\varepsilon_n}$. By (f_5), (u_n) is a bounded sequence. Hence it possesses a weakly convergent subsequence having limit u. By (f_2), $B(u_n) \to B(u)$ along this subsequence. Since Lu_n also converges weakly to Lu along the subsequence and $f'(u_n) \to 0$, it follows that $Lu_n = f'(u_n) - B(u_n) \to -B(u)$. Hence $Lu_n \to Lu$ and $f'(u) = 0$. By (f_2) again,

$$f(u_n) = \frac{1}{2}(Lu_n, u_n) + b(u_n) \to f(u) = c .$$

But then c is a critical value of f, contrary to assumption. Thus there exists an ε as desired above. It can further be assumed that $\varepsilon < 4^{-1}$.

Choose an $h \in \Lambda$ with a corresponding β such that

(1.30) $$c \leq \sup_{u \in \bar{Q}} f(h_1(u)) \leq c + \varepsilon$$

and $h_t(\partial Q) \subset G_{\frac{\alpha+\omega}{2}-\beta}$. By (Γ_4), $h_1(\overline{Q})$ is bounded. Therefore there is an $R > 0$ such that $h_1(\overline{Q}) \subset B_R$. By Theorem 1.5 with $\gamma = \frac{1}{2}\min(\beta, \varepsilon)$, there exists an $\eta \in \Gamma$ and $k \in N$ such that η_t^k satisfies 1^o and 2^o of that theorem. Let $g_t(u) = \eta_t^k(h_t(u))$. Since $h_t(\partial Q) \subset G_{\frac{\alpha+\omega}{2}-\beta} \cap B_R$, $g_t(\partial Q) \subset G_{\frac{\alpha+\omega}{2}-\frac{\beta}{2}}$. Hence $g \in \Lambda$ and by (1.16).

$$c \le \sup_{u \in \overline{Q}} f(g_1(u)). \tag{1.31}$$

We know from (1.30) that $f(h_1(u)) \le c + \varepsilon$ for all $u \in \overline{Q}$. Thus if $h_1(u) \in f^{-1}([c-\varepsilon, c+\varepsilon])$, then $g_1(u) \in G_{c-\frac{\varepsilon}{2}}$ via 2^o of Theorem 1.5. On the other hand if $h_1(u) \in G_{c-\varepsilon}$, then $g_1(u) \in G_{c-\frac{\varepsilon}{2}}$ by 1^o of Theorem 1.5 and our choice of γ. Consequently

$$\sup_{u \in \overline{Q}} f(g_1(u)) \le c - \frac{\varepsilon}{2} \tag{1.32}$$

contrary to (1.31) and the theorem is proved.

Proof of Theorem 1.4: Immediate from Theorem 1.20 since (f_3) implies (f_5) with any choice of $a > 0$.

Proof of Theorem 0.1: Immediate from Theorem 1.4 with $v = 0$ in (Γ_4) since (f_4) and Lemma 1.3 imply $(\overline{\Gamma}_4)$

We conclude this section with the subquadratic analogue of Theorem 0.1.

Theorem 1.33: Let the hypotheses of Theorem 0.1 be satisfied with (f_4) replaced by

(f_6) There are constants $R > 0$ and $\alpha > \omega$ and a $v \in Q \equiv B_R \cap E_2$ such that

(i) $f \ge \alpha$ on $S \equiv v + E_1$

(ii) $f \le \omega$ on ∂Q.

Then f possesses a critical value $c \ge \alpha$.

Proof: Immediate from Theorem 1.4 since (f_6) and Lemma 1.1 imply $(\overline{\Gamma}_4)$.

Remark 1.34: Theorem 1.33 can be generalized slightly by replacing Q by $\hat{Q} = e + Q$ where $e \in E_1$ and requiring that $f \le \omega$ on $\partial\hat{Q}$. The proof of Lemma 1.1 then shows S and $\partial\hat{Q}$ link so Theorem 1.4 is again applicable.

§2. An example.

In this section we give a simple model problem which illustrates the indefinite nature of functionals to which Theorem 0.1 can be applied.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. To avoid distinguishing between cases later, assume $n \geq 2$. In what follows we denote $(v, w) \in \mathbb{R}^2$ by u and $(v^2 + w^2)^{\frac{1}{2}}$ by $|u|$. Consider the elliptic system of partial differential equations:

$$(2.1) \qquad \begin{cases} -\Delta v = |u|^{s-1} v & x \in \Omega \\ -\Delta w = -|u|^{s-1} w & \\ v = w = 0, & x \in \partial\Omega \end{cases}$$

where $1 < s < (n+2)(n-2)^{-1}$. Let $W_0^{1,2}(\Omega)$ denote the space of functions having square integrable first derivatives in Ω and vanishing in a generalized sense on $\partial\Omega$. Let $E = (W_0^{1,2}(\Omega))^2$ with

$$\|u\|_E^2 \equiv \|u\|^2 = \int_\Omega |\nabla u|^2 \, dx .$$

For $u \in E$, set

$$(2.2) \qquad f(u) = \int_\Omega [\tfrac{1}{2}(|\nabla v|^2 - |\nabla w|^2) - \tfrac{1}{s+1} |u|^{s+1}] \, dx.$$

The indefinite nature of f is exceedingly clear.

We will show

Theorem 2.3: f as defined in (2.2) possesses a positive critical value in E.

Remark 2.4: The corresponding critical point u will be a weak solution of (2.1) and standard regularity arguments then show $u \in (C^2(\Omega))^2$. These regularity considerations will not be discussed further here. Note that 0 is a critical value of f so $c \neq 0$ implies the corresponding critical point is a nontrivial solution of (2.1). If $|u|^{s+1}$ is replaced by $|u|^{s+1} + g(x, u)$ in (2.2) where $g \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ and satisfies

(g_1) $\quad g(x,u) \geq 0$ in $\Omega \times \mathbb{R}^2$

(g_2) $\quad g(x,u) = o(|u|^2)$ at $u = 0$

(g_3) $\quad |g_u(x,u)| \leq a_1 + a_2 |u|^\sigma$ for all $u \in \mathbb{R}^2$ with $1 \leq \sigma < s$,

the same theorem obtains with the critical point u a classical solution of (2.1). However we prefer to keep this example as simple as possible. In (g_3) and elsewhere in this paper, a_i and M_j, $i, j \in \mathbb{N}$ repeatedly denote nonnegative constants.

Proof of Theorem 2.3: Setting $E_1 = W_0^{1,2}(\Omega) \times \{0\}$, $E_2 = \{0\} \times W_0^{1,2}(\Omega)$ $u_1 = (v, 0)$, $u_2 = (0, w)$, $L_i u_i = (-1)^{i-1} u_i$, $i = 1, 2$, and

$$b(u) = -\int_\Omega \frac{1}{s+1} |u|^{s+1} \, dx$$

for $u \in E$, we see that $E_1 = E_2^{\perp}$ and f satisfies (f_1). That (f_2) is satisfied and $f \in C^1$ follows readily from the form of b and the fact that E is compactly embedded in $(L^t(\Omega))^2$ for $1 \leq t < \frac{2n}{n-2}$. Since a more complicated such result will be proved in §3, we omit further

details here. (See also [7]).

It remains to verify (f_3) and (f_4). Suppose $f(u_m) \le M$ and $f'(u_m) \to 0$ as $m \to \infty$. Then for large m,

(2.5) $$|f'(u_m)\varphi| \le \|\varphi\|$$

for all $\varphi \in E$. Choosing $\varphi = u_m$ gives

(2.6) $$M + \frac{1}{2}\|u_m\| \ge f(u_m) - \frac{1}{2} f'(u_m)u_m = \int_\Omega \left(\frac{1}{2} - \frac{1}{s+1}\right)|u_m|^{s+1}\,dx .$$

Next successively choosing $\varphi = (v_m, 0)$ and $\varphi = -(0, w_m)$ in (2.5) and adding yields:

(2.7) $$\|u_m\|^2 \le a_3 \int_\Omega |u_m|^{s+1}\,dx + 2\|u_m\| .$$

Combining (2.6) and (2.7) gives

(2.8) $$\|u_m\|^2 \le a_4\|u_m\| + a_5$$

from which the boundedness of (u_m) and (f_3) follow.

Next suppose $u = (v, 0) \in E_1$. Since

(2.9) $$|b(u)| \le a_6\|u\|^{s+1}$$

via the Sobolev embedding theorem,

(2.10) $$f(u) \ge \frac{1}{2}\|u\|^2 - a_6\|u\|^{s+1} = \frac{1}{2}\|u\|^2\,(1 - 2a_6\|u\|^{s-1}).$$

Choosing ρ so that $1 = 4a_6\rho^{s-1}$, we have

(2.11) $$f(u) \ge \frac{1}{4}\rho^2 \equiv \alpha$$

on $S = \partial B_\rho \cap E_1$, i.e. $(f_4)(i)$ holds. Lastly to obtain $(f_4)(ii)$,

consider $f(u + re)$ where $r \ge 0$, $e = (\hat{v}, 0) \in \partial B_1$, $u = (0, w) \in B_{r_2} \cap E_2$, and r_2 is free for the moment. Thus

(2.12) $$f(u + re) = \frac{r^2}{2} - \|u\|^2 - \frac{1}{s+1}\int_\Omega |u + re|^{s+1}\,dx$$

so $f \le 0$ if $r = 0$. By the Hölder inequality,

(2.13) $$\frac{1}{s+1}\int_\Omega |u + re|^{s+1}\,dx \ge a_7\left(\int_\Omega |u + re|^2\,dx\right)^{\frac{s+1}{2}}$$
$$= a_7\left(\int_\Omega (|u|^2 + r^2|e|^2)\,dx\right)^{\frac{s+1}{2}} .$$

Hence by (2.12) - (2.13),

$$f(u + re) \le \frac{r^2}{2} - \|u\|^2 - a_7 r^{s+1}\left(\int_\Omega |e|^2\,dx\right)^{\frac{s+1}{2}}.$$

Choose r_1 so that

(2.14) $$\varphi(r) \equiv \frac{r^2}{2} - a_7 r^{s+1}\left(\int_\Omega |e|^2\,dx\right)^{\frac{s+1}{2}} \le 0$$

for $r \ge r_1$. If $M = \max_{r \in [0, r_1]} \varphi(r)$, choose r_2 so that $M \le r_2^2$. It then follows from (2.12) - (2.14) that $f \le 0 \equiv \omega$ on ∂Q and the proof of Theorem 2.3 is complete.

§3. Applications to superquadratic Hamiltonian systems

This and the following section contain applications of the results of §1 to Hamiltonian systems of ordinary differential equations. We treat the superquadratic case now and the subquadratic case in §4. We will be concerned in particular with the existence of periodic solutions of such systems.

Let $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and consider the Hamiltonian system of ordinary differential equations:

(3.1) $$\dot{p} = -H_q(p,q), \quad \dot{q} = H_p(p,q)$$

where p and q are n-tuples and $\cdot$ denotes $\frac{d}{dt}$. Letting $z = (p,q)$ and $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ where I denotes the identity matrix in $\mathbb{R}^n$, (3.1) becomes

(3.2) $$\dot{z} = \mathcal{J} H_z(z)$$

As our first result for (3.2) we apply Theorem 0.1 to give a simplified proof of Theorem 2.1 of [2]. Below $(\cdot,\cdot)_{\mathbb{R}^j}$ denotes the usual inner product in $\mathbb{R}^j$.

<u>Theorem 3.3:</u> Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_1) $$H(z) \geq 0 \quad \text{for} \quad z \in \mathbb{R}^{2n},$$

(H_2) $$H(z) = o(|z|^2) \quad \text{at} \quad z = 0,$$

(H_3) There is a $\theta \in (0, \frac{1}{2})$ and $\overline{r} > 0$ such that
$$0 < H(z) \leq \theta(z, H_z(z))_{\mathbb{R}^{2n}} \quad \text{for} \quad |z| \geq \overline{r}.$$

Then for any $T > 0$, (3.2) possesses a nonconstant T periodic solution.

Before applying Theorem 0.1 to obtain Theorem 3.3, a few remarks and preliminaries are necessary. By making the change of variables $t \to 2\pi t T^{-1} \equiv \lambda^{-1} t$, (3.2) transforms to

(3.4) $$\dot{z} = \lambda \mathcal{J} H_z(z)$$

and we seek a 2π periodic solution of (3.4) with λ prescribed. Next, integrating (H_3) shows

(3.5) $$H(z) \geq a_1 |z|^{\frac{1}{\theta}} - a_2$$

for all $z \in \mathbb{R}^{2n}$. Hence H is a superquadratic nonlinearity. The lack of an upper bound for the rate at which $H(z) \to \infty$ as $|z| \to \infty$ creates technical difficulties which can be avoided by replacing (3.4) by a modified problem as follows. Let $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ if $s \leq K$, $\chi(s) = 0$ if $s \geq K+1$, and $\chi'(s) < 0$ if $s \in (K, K+1)$. Set

$$H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|)) a |z|^4$$

where $a = a(K) > 0$. Then $H_K \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and it is easy to verify (see e.g. [2]) that for a sufficiently large, H_K satisfies (H_1) - (H_3) with θ replaced by $\mu = \max(\theta, \frac{1}{4})$. Moreover $H_K(z)$ grows at a controlled superquadratic rate as $|z| \to \infty$.

Consider the new Hamiltonian system

(3.6) $$\dot{z} = \lambda \mathcal{J} H_{Kz}(z).$$

If (3.6) has a nonconstant 2π periodic solution z_K such that

(3.7) $$\| z_K \|_{L^\infty} \leq K$$

where L^∞ refers to $(L^\infty(S^1))^{2n}$, then z_K satisfies (3.4). Ultimately K will be chosen so as to satisfy (3.7).

The space in which we will treat (3.6) is E, the space of 2n-tuples of 2π periodic functions which possess a derivative of order $\frac{1}{2}$. Perhaps the simplest way to introduce this space is as follows. Let $z \in C^\infty(\mathbb{R}, \mathbb{R}^{2n})$ be 2π periodic. Then z has a Fourier expansion $z = \sum_{-\infty}^{\infty} a_k e^{ikt}$ with $a_k \in \mathbb{C}^{2n}$ and $a_{-k} = \bar{a}_k$. E is the closure of the set of such functions under the (Hilbert space) norm

$$\Big(\sum_{k \in \mathbb{Z}} (1+|k|)|a_k|^2\Big)^{\frac{1}{2}}.$$

E can also be identified with Sobolev space $(W^{\frac{1}{2},2}(S^1))^{2n}$ obtained by interpolating between $(L^2(S^1))^{2n}$ and $(W^{1,2}(S^1))^{2n}$. For our later purposes, it is convenient to put another norm on E equivalent to the above one. For $z \in E$, let p and q denote respectively the first and last n components of E. If z is smooth, the action integral of z is defined by

$$A(z) \equiv \int_0^{2\pi} (p, \dot{q})_{\mathbb{R}^n}\, dt.$$

Writing $p = \sum_{-\infty}^{\infty} p_j e^{ijt}$, $q = \sum_{-\infty}^{\infty} q_j e^{ijt}$,

$$A(z) = 2\pi \sum_{k \in \mathbb{Z}} k(p_k, q_k)_{\mathbb{C}^n}$$

so in fact A extends to a continuous quadratic form on E. It is easy to verify – see e.g. [2] – that if E^0, E^+, E^- are the (closed) subspaces of E on which A is null, positive definite, and negative definite,

$E = E^0 \oplus E^+ \oplus E^-$. Moreover if $z = (p, q)$ and $\zeta = (\varphi, \psi)$ belong to distinct such subspaces,

$$\int_0^{2\pi} \big((p, \dot{\psi})_{\mathbb{R}^n} + (\varphi, \dot{q})_{\mathbb{R}^n}\big)\, dt = 0$$

and

$$\int_0^{2\pi} (z, \zeta)_{\mathbb{R}^{2n}}\, dt = 0 .$$

Hence for $z = z^0 + z^+ + z^- \in E$, we can and will take as norm in E:

$$\|z\|_E^2 = |z^0|^2 + A(z^+) - A(z^-) \equiv \|z\|^2$$

and the associated inner product makes E a Hilbert space with E^0, E^+, E^- orthogonal subspaces of E. Finally we note that for any $\beta \in [1, \infty)$ and $z \in (L^\beta(S^1))^{2n}$,

$$\|z\|_{L^\beta} \le a_\beta \|z\| \tag{3.8}$$

with the embedding of E in L^β being compact [3, 14].

Let

$$f(z) \equiv A(z) - \lambda \int_0^{2\pi} H_K(z)\, dt.$$

The definition of H_K and (3.8) imply that f is defined for $z \in E$. Moreover f is indefinite via our above discussion of $A(z)$. We will show that f satisfies the hypotheses of Theorem 0.1 (with $\omega = 0$). Let $E_1 = E^+$ and $E_2 = E^0 \oplus E^-$. Then $E_1 = E_2^\perp$ and f satisfies (f_1) with $L_1 z$ being defined for $z \in E_1$ by

$$(L_1 z, \zeta) = A'(z)\zeta \tag{3.9}$$

for all $\zeta \in E_i$, $i = 1, 2$. (In (3.9), $A'(z)\zeta$ denotes the Frechet derivative of A at z acting on ζ). Hence

$$\frac{1}{2}(Lz, z) = A(z) \quad \text{and} \quad b(z) = -\lambda \int_0^{2\pi} H_K(z)\, dt.$$

It remains to prove that $f \in C^1(E, \mathbb{R})$ and satisfies $(f_2) - (f_4)$. The verification of the smoothness of f follows similar lines as does (f_2) but is simpler. Hence we will prove the latter and leave the former to the reader.

__Lemma 3.10__ If $H \in C(\mathbb{R}^{2n}, \mathbb{R})$, b is weakly continuous.

Proof: Since H is continuous, so is H_K. Let $(z_m) \subset E$ and suppose z_m converges weakly to $\hat{z}$. Then by (3.8) and the remark following it, $z_m \to \hat{z}$ in L^β for all $\beta \in [1, \infty)$. A theorem of Krasnoselski [5] implies that if $g \in C(S^1 \times \mathbb{R}^{2n}, \mathbb{R})$ and satisfies

$$(3.11) \qquad |g(t, z)| \le a_3 + a_4 (z)^{\gamma/\delta},$$

then

$$\int_0^{2\pi} g(t, z(t))\, dt$$

is continuous from $(L^\gamma(S^1))^{2n}$ to $L^\delta(S^1)$ for $1 \le \gamma, \delta < \infty$. The definition of H_K shows that it satisfies (3.11) with $\gamma = 4$ and $\delta = 1$. Hence choosing $\beta = 4$, we see $z_m \to \hat{z}$ in L^4 and therefore $b(z_m) \to b(\hat{z})$.

Next we present a result which implies the uniform differentiability of b and is also useful in other contexts.

__Proposition 3.12__ Let $\Omega \subset \mathbb{R}^j$ be a bounded domain and for some $s, \hat{s}$, $g \in C^1(\Omega \times \mathbb{R}^k, \mathbb{R})$ where $g = g(x, z)$ and

$$(g_1) \qquad |g_z(x, z)| \le a_1 + a_2 |z|^s, \quad 1 \le s < \hat{s}.$$

If $\hat{E}$ is a subspace of $(L^1(\Omega))^k$ with

$$(3.13) \qquad \|z\|_{L^r} \le a_r \|z\|_{\hat{E}}$$

for all $z \in \hat{E}$ and all $r \in [2, \hat{s} + 1]$, then

$$b(z) = \int_\Omega g(x, z(x))\, dx$$

is uniformly differentiable on bounded subsets of $\hat{E}$.

Proof: For notational convenience the x dependence of g will be suppressed in what follows. The verification that $b(z)$, $b'(z)$ are defined for $z \in \hat{E}$ will be left to the reader.

Choose ε and $R > 0$ and suppose that $\|z\|_{\hat{E}}$, $\|z + h\|_{\hat{E}} \le R$. We must find a $\delta = \delta(\varepsilon, R)$ such that

$$(3.14) \qquad |b(z + h) - b(z) - b'(z)h| \le \varepsilon \|h\|_{\hat{E}}$$

whenever $\|h\|_{\hat{E}} \le \delta$. Observe that

$$(3.15) \qquad |b(z+h) - b(z) - b'(z)h| \le \int_\Omega |g(z+h) - g(z) - g_z(z)h|\, dx$$

$$\equiv \int_\Omega G(x, z)\, dx.$$

Let $I = \{x \in \overline{\Omega} \mid |z(x)| \ge \beta\}$

$II = \{x \in \overline{\Omega} \mid |h(x)| \ge \gamma\}$

$III = \{x \in \overline{\Omega} \mid |h(x)| \le \gamma \text{ and } |z(x)| \le \beta\}$

where β and γ are free for the moment. The right hand side of (3.15) is dominated by the sum of the integrals of G over these three sets. These integrals will be estimated next. First

$$(3.16) \qquad \int_I G\,dx \le \int_I (|g(z+h) - g(z)| + |g_z(z)h|)\,dx .$$

Hence by the mean value theorem, (g_1), and the Hölder inequality,

$$(3.17) \qquad \int_I G\,dx \le a_3\left(\int_I 1^\sigma dx\right)^{1/\sigma} \left[\left(\int_I |z|^{s+1} dx\right)^{\frac{s}{s+1}} + \left(\int_I |h|^{s+1} dx\right)^{\frac{s}{s+1}}\right] \left(\int_I |h|^{\hat{s}+1} dx\right)^{\frac{1}{\hat{s}+1}}$$

where $\sigma \in (1,\infty)$ is defined by

$$\frac{1}{\sigma} + \frac{s}{s+1} + \frac{1}{\hat{s}+1} = 1$$

and such a σ exists since $s < \hat{s}$. Hence by (3.13),

$$(3.18) \qquad \int_I G\,dx \le a_4 M^{1/\sigma}(R^s + \|h\|^s_{\hat{E}}) \quad \|h\|_{\hat{E}}$$

where M denotes the measure of I. Since

$$R \ge \|z\|_{\hat{E}} \ge a_5 \|z\|_{L^2} \ge a_5\left(\int_I |z|^2 dx\right)^{\frac{1}{2}} \ge a_5 \beta M^{\frac{1}{2}},$$

we see

$$M^{1/\sigma} \le \left(\frac{R}{a_5\beta}\right)^{2/\sigma} \equiv M_1$$

and (3.18) becomes

$$(3.19) \qquad \int_I G\,dx \le a_4 M_1 \,(R^s + \|h\|^s_{\hat{E}})\, \|h\|_{\hat{E}} .$$

Next as in (3.16) - (3.17),

$$(3.20) \qquad \int_{II} G\,dx \le a_6\left[\left(\int_{II} |z|^{s+1} dx\right)^{\frac{s}{s+1}} + \left(\int_{II} |h|^{s+1} dx\right)^{\frac{s}{s+1}}\right] \gamma^{1-\alpha}\left(\int_{II} |h|^{\hat{s}+1} dx\right)^{\frac{1}{s+1}}$$

$$\le a_7[R^s + \|h\|^s_{\hat{E}}] \quad \|h\|^\alpha_{\hat{E}}$$

where $\alpha = (\hat{s}+1)(s+1)^{-1} > 1$ and a_7 depends on γ.

Finally to estimate the remaining term, observe that since $g \in C^1$, given any $\hat{\varepsilon}, \hat{\beta} > 0$, there is a $\hat{\gamma} = \hat{\gamma}(\hat{\varepsilon}, \hat{\beta})$ such that $|z| \le \hat{\beta}$ and $|h| \le \hat{\gamma}$ imply that

$$(3.21) \qquad |g(z+h) - g(z) - g_z(z)h| \le \hat{\varepsilon}|h| .$$

Hence if $\beta \le \hat{\beta}$ and $\gamma \le \hat{\gamma}$, (3.21) shows

$$(3.22) \qquad \int_{III} G\,dx \le \hat{\varepsilon} \int_{III} |h|\,dx \le a_8\,\hat{\varepsilon}\, \|h\|_{\hat{E}}$$

via (3.13) and the Hölder inequality.

Now choose $\hat{\varepsilon}$ so that $a_8\hat{\varepsilon} \le \frac{\varepsilon}{3}$ and choose $\hat{\beta} = \beta$. This determines $\hat{\gamma}$ in terms of ε and β. Combining (3.19), (3.20), and (3.22) yields

$$(3.23) \qquad \int_\Omega G\,dx \le a_4 M_1 \,(R^s + \|h\|^s_{\hat{E}})\, \|h\|_{\hat{E}} + a_7(R^s + \|h\|^s_{\hat{E}})\|h\|^\alpha_{\hat{E}} + \frac{\varepsilon}{3}\|h$$

The proof is now completed by choosing β so large that $a_4 M_1 R^s \le \frac{\varepsilon}{3}$, taking $\gamma = \hat{\gamma}$ and δ so small that $a_4 M_1 \delta^s + a_7(R^s + \delta^s)\delta^{\alpha-1} \le \varepsilon/3$.

<u>Corollary 3.24:</u> If $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, b is uniformly differentiable on bounded subsets of E.

Proof: Immediate from Proposition 3.12, the definition of H_K, and (3.8).

Lemma 3.25: If H satisfies (H_3), f satisfies (f_3).

Proof: As was noted earlier, if H satisfies (H_3), so does (H_K) with θ replaced by $\mu = \max(\theta, \frac{1}{4})$. Suppose (z_m) is a sequence with $f(z_m) \le M$ and $f'(z_m) \to 0$ as $m \to \infty$. Then for large m with $z = z_m$ we have

$$M + \frac{1}{2}\|z\| \ge f(z) - \frac{1}{2}f'(z)z = \lambda\int_0^{2\pi}[\frac{1}{2}(z, H_{Kz}(z))_{R^{2n}} - H_K(z)]\, dt. \tag{3.26}$$

Applying (H_3) for (H_K) gives

$$M + \frac{1}{2}\|z\| \ge \lambda(\frac{1}{2} - \mu)\int_0^{2\pi}(z, H_{Kz}(z))\, dt - M_1 \tag{3.27}$$

where the constant M_1 is independent of K. The form of H_K then implies

$$\|z\|_{L^4}^4 \le M_2 + M_3\|z\| \tag{3.28}$$

where M_2 and M_3 depend on K (as well as λ and μ). Moreover by (3.27), (H_3), (3.5) for H_K, and the Hölder inequality,

$$1 + \|z\| \ge M_4\int_0^{2\pi}|z|^{\frac{1}{\mu}}\, dt \ge M_5(\int_0^{2\pi}|z|^2\, dt)^{\frac{1}{2\mu}}. \tag{3.29}$$

Writing $z = z^0 + z^+ + z^- \in E^0 \oplus E^+ \oplus E^-$, substituting in the right hand side of (3.29) and using the L^2 orthogonality of the components of z yields

$$|z^0| \le M_6(1 + \|z\|^{\mu}). \tag{3.30}$$

Moreover since

$$|f'(z_m)\zeta| \le \|\zeta\| \tag{3.31}$$

for large m and all $\zeta \in E$, choosing $\zeta = z_m^+$ gives

$$2\|z^+\|^2 = A'(z)z^+ \le \lambda\int_0^{2\pi}|H_{Kz}(z)|\,|z^+|\, dt + \|z^+\| \tag{3.32}$$

$$\le M_7(\|z\|_{L^4}^3 + 1)\,\|z^+\|_{L^4} + \|z^+\|.$$

Hence by (3.32), (3.8), and (3.28),

$$\|z^+\| \le M_8(1 + \|z\|^{3/4}). \tag{3.33}$$

Taking $\zeta = -z_m^-$ in (3.31) yields a similar inequality for z^-. Adding the two to (3.30) shows

$$\|z\| \le M_9(1 + \|z\|^{3/4} + \|z\|^{\mu}) \tag{3.34}$$

from which the boundedness of (z_m) follows.

Next we will verify (f_4).

Lemma 3.35: If H satisfies (H_2), f satisfies (f_3) (i).

Proof: By (H_2), for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|H(z)| = |H_K(z)| \le \varepsilon|z|^2$$

if $|z| \le \delta < K$. The form of H_K then implies there is a constant $A_\varepsilon > 0$ so that

$$|H_K(z)| \le \varepsilon|z|^2 + A_\varepsilon|z|^4 \tag{3.36}$$

for all $z \in R^{2n}$. Consequently for $z \in E$, (3.36) and (3.8) show that

$$\int_0^{2\pi}|H_K(z)|\,dt \le \varepsilon\|z\|_{L^2}^2 + A_\varepsilon\|z\|_{L^4}^4 \le a_3(\varepsilon\|z\|^2 + A_\varepsilon\|z\|^4). \tag{3.37}$$

Thus for $z \in E_1$,

$$f(z) \geq \|z\|^2 - \lambda a_3(\varepsilon\|z\|^2 + A_\varepsilon\|z\|^4) \tag{3.38}$$
$$= \|z\|^2(1 - \lambda a_3\varepsilon - \lambda a_3 A_\varepsilon\|z\|^2).$$

Choosing $\varepsilon = (3\lambda a_3)^{-1}$ and ρ so that $\lambda a_3 A_\varepsilon \rho^2 \leq 3^{-1}$ gives

$$f(z) \geq \frac{1}{3}\rho^2 \equiv \alpha \tag{3.39}$$

on $S = \partial B_\rho \cap E_1$.

Remark 3.40: Note that ρ and α depend on K via A_ε.

Lemma 3.41: If H satisfies (H_1) and (H_3), there exists a set Q with r_1 and r_2 independent of K such that f satisfies (f_3)(ii) (with $\omega = 0$).

Proof: Let $z = z^0 + z^- \in B_{r_2} \cap E_2$ where r_2 is free for now and consider

$$f(z + re) = r^2 - \|z^-\|^2 - \lambda\int_0^{2\pi} H_K(z + re)\,dt. \tag{3.42}$$

Note that for $r = 0$, $f(z) \leq 0$ via (H_1). By (3.5) for (H_K) and the Hölder inequality,

$$\int_0^{2\pi} H_K(z+re)\,dt \geq a_3\left(\int_0^{2\pi} |z + re|^2\,dt\right)^{\frac{1}{2\mu}} - a_4 \geq \tag{3.43}$$
$$\geq a_3\left(\int_0^{2\pi}(|z^0|^2 + r^2|e|^2)\,dt\right)^{\frac{1}{2\mu}} - a_4$$
$$\geq a_5\left[|z^0|^{\frac{1}{\mu}} + r^{\frac{1}{\mu}}\left(\int_0^{2\pi}|e|^2\,dt\right)^{\frac{1}{2\mu}}\right] - a_4$$

as in (3.29). Hence by (3.42) - (3.43),

$$f(z + re) \leq r^2 - \|z^-\|^2 - a_6(r^{\frac{1}{\mu}} + |z^0|^{\frac{1}{\mu}}) + a_7. \tag{3.44}$$

Choose r_1 so that

$$\varphi(r) \equiv r^2 - a_6 r^{\frac{1}{\mu}} + a_7 \leq 0 \tag{3.45}$$

for $r \geq r_1$. Let $M = \max_{r\in[0,r_1]} \varphi(r)$. Since

$$\|z^-\|^2 + a_6|z^0|^{1/\mu} \geq a_8 \min(\|z\|^2, \|z\|^{1/\mu}),$$

by choosing r_2 large enough we have

$$M \leq \|z^-\|^2 + a_6|z^0|^{\frac{1}{\mu}} \tag{3.46}$$

for $\|z\| \geq r_2$. It then follows from (3.44) - (3.46) and (H_1) that $f \leq 0 \equiv \omega$ on ∂Q. Note also that r_1 and r_2 are independent of K.

Proof of Theorem 3.3: The above lemmas show f satisfies the hypotheses of Theorem 0.1. Therefore f has a positive critical value c and corresponding critical point z. It remains to show that $z = z(K)$ is a nonconstant solution of (3.6) and that $\|z\|_{L^\infty} \leq K$ for appropriately chosen K. Proof of the first statement requires a simple regularity argument. Since $z = (p, q)$ is a critical point of f,

$$f'(z)\zeta = 0 \tag{3.47}$$

for all $\zeta = (\varphi, \psi) \in E$. For smooth ζ, (3.47) implies that

$$\int_0^{2\pi}[(p, \dot\psi)_{\mathbb{R}^n} - (q, \dot\varphi)_{\mathbb{R}^n}]\,dt = \lambda\int_0^{2\pi}(H_{Kz}(z), \zeta)_{\mathbb{R}^{2n}}\,dt. \tag{3.48}$$

For $w \in (L^2(S^1))^j$, let

$$[w] = \frac{1}{2\pi} \int_0^{2\pi} w(t)\,dt.$$

i.e. $[w]$ is the mean value of w. By (3.8) and the form of H_K, $H_{Kz}(z) \in (L^2(S^1))^{2n}$ and (3.48) with $\zeta = 1$ shows that $[H_{Kz}(z)] = 0$. Observe also that if $y \in L^2(S^1)$ and $[y] = 0$, there exists an $x \in W^{1,2}(S^1)$ such that $\dot{x} = y$ and x is uniquely determined if its mean value is specified. Thus we can find a unique $\hat{z} = (\hat{p}, \hat{q}) \in (W^{1,2}(S^1))^{2n}$ such that $[\hat{z}] = [z]$ and

(3.49) $$\dot{\hat{z}} = \lambda \mathscr{J} H_{Kz}(z).$$

Taking the inner product of (3.49) with $\mathscr{J}\zeta$ where $\zeta \in E$ is smooth yields

(3.50) $$\int_0^{2\pi} [-(\psi, \dot{\hat{p}})_{\mathbb{R}^n} + (\varphi, \dot{\hat{q}})_{\mathbb{R}^n}]\,dt = \lambda \int_0^{2\pi} (H_{Kz}(z), \zeta)_{\mathbb{R}^{2n}}\,dt$$
$$= \int_0^{2\pi} [(\hat{p}, \dot{\psi})_{\mathbb{R}^n} - (\hat{q}, \dot{\varphi})_{\mathbb{R}^n}]\,dt.$$

Comparing (3.50) to (3.48) and noting that $[z] = [\hat{z}]$ and smooth ζ are dense in E shows that $z = \hat{z}$ ae. and $z \in (W^{1,2}(S^1))^{2n}$. Hence by (3.48), (3.6) is satisfied in an a.e. sense. But since $z \in (W^{1,2}(S^1))^{2n}$, the Sobolev embedding theorem implies z and $H_{Kz}(z)$ are continuous. Thus (3.6) implies $z \in C^1$ and we have a classical solution of the equation. Moreover z is nonconstant since $f(z) = c \geq \alpha > 0$ by Theorem 0.1 while if z were constant,

$$f(z) = -\lambda \int_0^{2\pi} H_K(z)\,dt \leq 0$$

via (H_1).

Lastly to obtain the L^∞ upper bound for z, recall that by (1.16),

$$c = \inf_{h \in \Lambda} \max_{z \in \overline{Q}} f(h_1(z)).$$

Since $h(t, z) \equiv z \in \Lambda$,

$$c \leq \max_{z \in \overline{Q}} f(z).$$

Writing $z = z^0 + z^- + re$ for $z \in \overline{Q}$ where $r \in [0, r_1]$ and $\|z^0 + z^-\| \leq r_2$, we have

$$f(z) \leq r^2 - \|z^-\|^2 - \lambda \int_0^{2\pi} H_K(z)\,dt \leq r_1^2.$$

Hence $c \leq r_1^2$ and by Lemma 3.41, this estimate is independent of K. Now (3.26) - (3.27) with M replaced by $f(z) = c$ and the $\|z\|$ term omitted because of (3.47) give

(3.51) $$r_1^2 + M_1 \geq c + M_1 \geq \lambda(\tfrac{1}{2} - \mu) \int_0^{2\pi} (z, H_{Kz}(z))_{\mathbb{R}^{2n}}\,dt$$

where M_1 is independent of K. By (H_3),

(3.52) $$H_K(\zeta) \leq \mu(\zeta, H_{Kz}(\zeta))_{\mathbb{R}^{2n}} + M_2$$

for all $\zeta \in \mathbb{R}^{2n}$. Letting $\zeta = z(t)$, integrating (3.52) over $[0, 2\pi]$ and using the fact that $H_K(z) \equiv$ constant since (3.6) is a Hamiltonian system yields

(3.53) $$2\pi\, H_K(z) \leq \mu \int_0^{2\pi} (z, H_{Kz}(z))_{\mathbb{R}^{2n}} + 2\pi\, M_2$$

Now combining (3.51), (3.53), and (3.5) for (H_K) gives the desired L^∞ estimate for z and the proof of Theorem 3.3 is complete.

Remark 3.53': If H depends explicitly on t in a T periodic fashion and satisfies (H_1)-(H_3), Theorem 0.1 can again be applied to get a T periodic solution of (3.6). A further growth condition on H_z such as was used in [2] is then required to go from (3.6) to (3.4).

Corollary 3.54: Suppose $H(z) = Q(z) + \hat{H}(z) \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ where Q is a quadratic form and $\hat{H}$ satisfies (H_1)-(H_3). Then the conclusion of Theorem 3.3 obtains.

Proof: For $z \in E$, let

$$\hat{A}(z) = A(z) - \lambda \int_0^{2\pi} Q(z)dt$$

where A, λ are as earlier. If $L, \hat{L}$ are respectively the (bounded) self adjoint linear operators associated with A and $\hat{A}$, i.e.

$$(Lz,z) = A(z) \ , \ (\hat{L}z,z) = \hat{A}(z) \ ,$$

it is easily verified that $\hat{L} = L+C$ where C is compact. Therefore the essential spectrum of $\hat{L}$ is the same as that of L, namely $\{1,-1\}$. Hence $\ker \hat{L}$ is finite (and possibly zero) dimensional. Moreover E has a splitting into orthogonal subspaces $\hat{E}^+ \oplus \hat{E}^- \oplus \hat{E}^0$ with

$$\hat{A}(z^+) \geq \lambda^+ \|z^+\|_E^2$$
$$\hat{A}(z^-) \leq \lambda^- \|z^-\|_E^2$$

for $z^+ \in \hat{E}^+$, $z^- \in \hat{E}^-$, where λ^+, λ^- are respectively the smallest positive and largest negative eigenvalue of $\hat{L}$ and $\hat{E}^0 = \ker \hat{L}$. The proof of Theorem 3.3 now carries over to this situation with only minor changes.

Remark 3.54': A result similar to Corollary 3.54 with Q positive definite was obtained in [2]. A somewhat deeper result in which H is only required to satisfy (H_3) can be found in [3].

For our final two results for (3.2) in this section we prescribe the energy of the solution rather than the period and study the relationship between superquadratic growth at ∞ and at 0 and the resulting period.

Theorem 3.55: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies (H_3). Then for all large $a > 0$, (3.2) has a $T = T(a)$ periodic solution on $H^{-1}(a)$ and $T(a) \to 0$ as $a \to \infty$.

Proof: Observe first that (H_3) implies that for all large a, $H^{-1}(a)$ is a compact C^1 manifold in $\mathbb{R}^{2n}$ which bounds a star shaped region. A device from [3] will be employed. Suppose that $H^{-1}(a)$ is exterior to $B_{\overline{r}}$ in $\mathbb{R}^{2n}$ for all $a \geq a_0$. Fix such an a. Define a new function $\overline{H}(z)$ as follows: Let $\psi : S^{2n-1} \to H^{-1}(a)$ be the radial map and $\tau > 2$ be free for now. Let $\overline{H}(0) = 0$ and for $z \neq 0$,

$$\overline{H}(z) = a \frac{|z|^\tau}{|\psi(z/|z|)|^\tau} \ .$$

Since $\psi \in C^1$, $\overline{H} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and is homogeneous of order τ. Thus $\overline{H}$ satisfies the hypotheses of Theorem 3.3. By that theorem with $T = 2\pi$, (3.2) with H replaced by $\overline{H}$ possesses a nonconstant 2π periodic solution $w(t)$. Since (3.2) is a Hamiltonian system, $\overline{H}(w(t)) \equiv \text{constant} = K$. By the homogeneity of $\overline{H}$, $\overline{H}(\beta w) = a$ where $\beta = (K^{-1}a)^{1/\tau}$. Then $\zeta = \beta w$ satisfies

(3.56) $$\dot{\zeta} = \beta \mathcal{J}\overline{H}_z(w) = \beta^{2-\tau} \mathcal{J}\overline{H}_z(\zeta) .$$

Thus ζ is a 2π periodic solution of (3.4) with $\lambda = \beta^{2-\tau}$ and H replaced by $\overline{H}$, and ζ lies on $H^{-1}(a)$. By reparametrizing ζ, we obtain a periodic solution of (3.2) on $H^{-1}(a)$. To see this, observe that since $H^{-1}(a)$ is a common level set for H and $\overline{H}$,

(3.57) $$\overline{H}_z(\xi) = \sigma(\xi)\, H_z(\xi)$$

for all $\xi \in H^{-1}(a)$ and $\sigma(\xi) \neq 0$ on this set. Consider $z(t) = \zeta(\delta(t))$. Then z satisfies

(3.58) $$\dot{z} = \beta^{2-\tau}\, \sigma(z)\, \mathcal{J} H_z(z)\dot{\delta} .$$

Choosing $\delta \in C^1$ so that $\delta(0) = 0$ and

(3.59) $$\dot{\delta} = \beta^{\tau-2}\, \sigma(\zeta(\delta(t)))^{-1},$$

z is a solution of (3.2) on $H^{-1}(a)$. Since the right hand side of (3.59) is positive, δ is monotone increasing and there is a $T = T(a)$ such that $\delta(T) = 2\pi$. Hence z is 2π periodic in t.

It remains to estimate T. By (3.57) and (H_3),

(3.60) $$\sigma(\xi) = \frac{\tau a}{(\xi, H_z(\xi))_{\mathbb{R}^{2n}}} \le \theta\tau$$

for $\xi \in H^{-1}(a)$. Integrating (3.59) then gives

(3.61) $$2\pi = \int_0^T \dot{\delta}\, dt \ge T(\theta\tau)^{-1} \beta^{\tau-2} = T(\theta\tau)^{-1} \left(\frac{a}{K}\right)^{\frac{\tau-2}{\tau}} .$$

To get a bound for K, recall w is a critical point of f so

(3.62) $$\begin{aligned} c &= f(w) - \tfrac{1}{2} f'(w)w \\ &= \int_0^{2\pi} [\tfrac{1}{2}(w, \overline{H}_z(w))_{\mathbb{R}^{2n}} - \overline{H}(w)]\, dt \\ &= (\tfrac{\tau}{2} - 1) \int_0^{2\pi} \overline{H}(w)\, dt = 2\pi(\tfrac{\tau}{2} - 1)K \equiv a_3 K \end{aligned}$$

Thus by (3.61) - (3.62),

(3.63) $$T \le 2\pi\theta\tau\, a^{\frac{2-\tau}{\tau}} \left(\frac{c}{a_3}\right)^{\frac{\tau-2}{\tau}} \equiv a_4 \left(\frac{c}{a}\right)^{\frac{\tau-2}{\tau}} .$$

To finish, we need an upper bound for c. For $z \in \mathbb{R}^{2n}\setminus\{0\}$,

(3.64) $$\overline{H}(z) = |z|^\tau\, \overline{H}\left(\frac{z}{|z|}\right) \ge |z|^\tau \min_{\xi \in S^{2n-1}} \overline{H}(\xi) .$$

Since $\psi(\xi) = \xi\rho(\xi)$ where $\rho(\xi) > 0$, $a = \overline{H}(\rho(\xi)\xi) = \rho(\xi)^\tau\, \overline{H}(\xi)$ so by (3.64),

(3.65) $$H(z) \ge \frac{a|z|^\tau}{\max\limits_{S^{2n-1}} \rho(\xi)^\tau} .$$

For $|z| \ge \bar{r}$, $H(z) \ge a_5 |z|^{1/\theta}$ via (H_3). In particular,

(3.66) $$a = H(\rho(\xi)\xi) \ge a_5\, \rho(\xi)^{1/\theta} .$$

Hence $\rho(\xi) \le \left(\frac{a}{a_5}\right)^\theta$ and

(3.67) $$\overline{H}(z) \ge a\left(\frac{a}{a_5}\right)^{-\theta\tau} |z|^\tau \equiv a^{1-\theta\tau} a_6 |z|^\tau .$$

Choose $\tau \in (2, \theta^{-1})$ so $\theta\tau < 1$ and

(3.68) $$\overline{H}(z) \ge a_0^{1-\theta\tau} a_6 |z|^\tau \equiv a_7 |z|^\tau$$

for $a \ge a_0$ with a_7 independent of $a \ge a_0$. Thus $\overline{H}$ satisfies (3.5) with constants independent of a for $a \ge a_0$. The proof of

Lemma 3.41 shows that r_1 depends only on a_0 for $a \ge a_0$. Thus

$$c \le \sup_{Q} f(z) \le r_1^2 \tag{3.69}$$

as in the proof of Theorem 3.3. Consequently

$$T \le a_4 \left(\frac{r_1^2}{a}\right)^{\frac{\tau-2}{\tau}} \to 0 \tag{3.70}$$

as $a \to \infty$ and the proof is complete.

<u>Remark 3.71:</u> Theorem 3.55 can also be proved using Theorem 1.1 of [2] and the estimates contained there. However that theorem uses deeper symmetry properties of (3.2) than the structure we have exploited here.

The arguments used in proving Theorem 3.55 apply equally well to other super and subquadratic problems at 0 and ∞. E.g.

<u>Theorem 3.72:</u> Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_3') There is a $\underline{\theta} \in (0, \frac{1}{2})$ and $\underline{r} > 0$ such that

$$0 < \underline{\theta}(z, H_z(z))_{\mathbb{R}^{2n}} \le H(z) \quad \text{for} \quad |z| \le \underline{r}.$$

Then for all small $a > 0$, (3.2) has a $T = T(a)$ periodic solution on $H^{-1}(a)$ and $T(a) \to \infty$ as $a \to 0$.

Proof: By (H_3'), there is an $a_0 > 0$ such that $H^{-1}(a)$ is a compact C^1 manifold which bounds a star shaped neighborhood of 0 and lies in $B_{\underline{r}}$ in $\mathbb{R}^{2n}$ for $0 < a \le a_0$. Define $\overline{H}$ as in Theorem 3.55 and argue in exactly the same fashion up to (3.60) which is replaced by the reverse inequality via (H_3'). Reversing (3.61) and applying (3.62) yields

$$T \ge a_4 \left(\frac{c}{a}\right)^{\frac{\tau-2}{\tau}} . \tag{3.73}$$

Now we need a lower bound for c. By (H_3') and (3.64) - (3.66)

$$\overline{H}(z) \le a^{1-\theta\tau} a_6 |z|^\tau \le a_0^{1-\theta\tau} a_6 |z|^\tau \equiv a_7 |z|^\tau \tag{3.74}$$

for $a \le a_0$. Using this in (3.36), the proof of Lemma 3.35 gives α independent of a for $a \le a_0$. Hence

$$T \ge a_4 \left(\frac{\alpha}{a}\right)^{\frac{\tau-2}{\tau}} \to \infty \tag{3.75}$$

as $a \to 0$.

<u>Remark 3.76:</u> We have not shown that $T(a)$ is a minimal or primitive period for the corresponding solution z_a. Thus there is a $k = k(a) \in \mathbb{N}$ such that Tk^{-1} is the minimal period of z_a and it is conceivable that $Tk^{-1} \not\to \infty$ as $a \to 0$.

§4. Applications to subquadratic Hamiltonian systems

This section contains several applications of Theorem 1.33 to forced and autonomous Hamiltonian systems.

Theorem 4.1: Suppose $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_4) $H(t,z)$ is T periodic in t

(H_5) There is a constant $M > 0$ such that $|H_z(t,z)| \le M$ for all $t \in [0,T]$ and $z \in \mathbb{R}^{2n}$

(H_6) $H(t,z) \to \infty$ as $|z| \to \infty$ uniformly for $t \in [0,T]$.

Then

$$(4.2) \quad \dot{z} = \mathcal{J} H_z(t,z)$$

possesses a T periodic solution.

Proof: For convenience we take $T = 2\pi$. Let E be as in Theorem 3.3, $E_1 = E^0 \oplus E^-$, $E_2 = E^+$, and

$$g(z) = \int_0^{2\pi} H(t,z)\,dt - A(z).$$

Since (H_5) implies that $|H(t,z)| \le M_1 + M_2|z|$ for $z \in \mathbb{R}^{2n}$, it follows as in Theorem 3.3 that $g \in C^1(E, \mathbb{R})$ and satisfies $(f_1)-(f_2)$ with L_1, L_2 and b as earlier aside from a factor of (-1). To check (f_3), suppose $g(z_m) \le \overline{M}$ and $g'(z_m) \to 0$ as $m \to \infty$. Then dropping subscripts and writing $z = z^0 + z^+ + z^-$, we have

$$(4.3) \quad \|z^-\|^2 \le \left|\int_0^{2\pi} H_z(t,z) z^-\,dt\right| + \|z^-\| \le M_2\|z^-\|$$

as in Lemma 3.25 via (H_5) and (3.8). Hence (z_m^-) and similarly (z_m^+) are uniformly bounded. In order to get an estimate for (z_m^0), note by the bounds already established,

$$(4.4) \quad \overline{M} \ge g(z) = \int_0^{2\pi} H(t,z)\,dt + \|z^-\|^2 - \|z^+\|^2$$
$$\ge \int_0^{2\pi} H(t,z^0)\,dt + \int_0^{2\pi} (H(t,z) - H(t,z^0))\,dt - M_2$$
$$\ge \int_0^{2\pi} H(t,z^0)\,dt - M\int_0^{2\pi} |z^+ + z^-|\,dt - M_3 .$$

Therefore (4.4), (3.8), and (H_6) imply (z_m^0) are uniformly bounded and (f_3) is satisfied.

Lastly we verify (f_6). If $z = z^0 + z^- \in E_1$,

$$g(z) \ge \int_0^{2\pi} H(t,z^0) + \|z^-\|^2 - M_4\|z^-\|$$

as in (4.4). Hence g is bounded from below on $S \equiv E_1$ by some constan[t] α. Similarly for $z \in E_2$,

$$g(z) \le M_5\|z\| - \|z\|^2$$

so if $z \in \partial Q = \partial B_R \cap E_2$ with R sufficiently large, $g(z) \le \alpha - 1 \equiv \omega$. Hence Theorem 1.33 (with $v = 0$ in (f_6)) implies g has a critical value $c \ge \alpha$. As in the proof of Theorem 3.3, any corresponding critical point is a classical solution of (4.2).

Corollary 4.5: If (H_6) is replaced by: $H(t,z) \to -\infty$ as $|z| \to \infty$ uniformly in t, then the conclusion of Theorem 4.1 obtains.

Proof: Let $f(z) = -g(z)$, $E_1 = E^0 \oplus E^+$, $E_2 = E^-$, and argue as abo[ve]

<u>Remark 4.6 (i)</u>: Theorem 4.1 and Corollary 4.5 are the analogues for Hamiltonian systems of results obtained earlier for elliptic equations by Ahmad, Lazer, and Paul [11]. See also [10], [12], [13].

(ii) If H is independent of t, Theorem 4.1 and Corollary 4.5 are valid with any choice of T. However (H_5) implies that $H_z(z)$ has a zero in B_M in R^{2n}. Thus for these results to be of interest in the autonomous case, one must further show the solution is nonconstant. An example of such a result will be given next.

<u>Theorem 4.7</u>: Suppose $H \in C^1(R^{2n}, R)$, satisfies $(H_5)-(H_6)$ and

(H_7) $H(0) = 0$ and $|H_z(z)|$, $H(z) > 0$ for $z \neq 0$.

Then for all T sufficiently large, (3.2) possesses a nonconstant T periodic solution.

Proof: Let

$$g(z) = \lambda \int_0^{2\pi} H(z)\,dt - A(z).$$

It follows from $(H_5)-(H_6)$ and Theorem 4.1 that g has a critical value e and (3.2) a corresponding solution for any $\lambda (= (2\pi)^{-1} T)$. By (H_5) and (H_7), $H_z(0) = 0$ and 0 is the only constant solution of (3.2). To complete the proof it suffices to show that $c > 0$ if λ is sufficiently large. By Theorem 1.33 and (H_7),

(4.8) $$c \geq \alpha = \inf_S g(z) = 0$$

since $S = E_1 = E^0 \oplus E^-$. Thus a better estimate than (4.8) is necessary. Let $v \in Q$, $v \neq 0$. By Lemma 1.1, $v + S \equiv S_v$ and ∂Q link. If we can show that

(4.9) $$\bar{\alpha} = \inf_{S_v} g(z) > 0,$$

then by Proposition 1.17 with S_v replacing S, $c \geq \bar{\alpha}$ and we are through. To verify (4.9), note that if $z \in S_v$, $z = z^0 + z^- + v$. Three cases are considered:

<u>Case (i)</u>: $\|z^-\|^2 > \|v\|^2 + 1$

Then

$$g(z) = \lambda \int_0^{2\pi} H(z)\,dt + \|z^-\|^2 - \|v\|^2 \geq 1.$$

<u>Case (ii)</u>: $\|z^-\|^2 \leq \|v\|^2 + 1$ and $|z^0| > a$

Then by (H_5),

(4.10) $$\int_0^{2\pi} H(z^0 + z^- + v)\,dt \geq 2\pi H(z^0) - M_1 \|z^- + v\|$$

as in (4.4). For a sufficiently large, which choice we make,

$$\int_0^{2\pi} H(z^0 + z^- + v)\,dt \geq 1$$

by (H_6) and (4.10). Hence

$$g(z) \geq \lambda - \|v\|^2 \geq 1$$

if $\lambda \geq \|v\|^2 + 1$.

<u>Case (iii)</u>: $\|z^-\|^2 \leq \|v\|^2 + 1$ and $|z^0| \leq a$.

Let $\Omega = \{z \in S_v \mid \|z^-\|^2 \leq \|v\|^2 + 1$ and $|z^0| \leq a\}$. Then Ω is convex and weakly compact. Since $\int_0^{2\pi} H(z)\,dt$ is weakly continuous,

it achieves its infimum, σ, on Ω at $\hat{z} = \hat{z}_0 + \hat{z}^- + v$ and $\hat{z} \neq 0$ because $v \neq 0$. Moreover $\sigma > 0$ by (H_7). Therefore

$$g(z) \geq \lambda\sigma + \|z^-\|^2 - \|v\|^2 \geq 1$$

for $\lambda \geq \sigma^{-1}(\|v\|^2 + 1)$.

Combining the three cases, we get $\bar{\alpha} \geq 1$ and the proof is complete.

Our next result is of the same nature as Theorem 4.1. In a sense we weaken (H_5) but strengthen (H_6).

Theorem 4.11: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, satisfies (H_4), and

(H_8) There is an $\bar{r} > 0$ and $\theta \in (\frac{1}{2}, 1)$ such that

$$H(t,z) \geq \theta(z, H_z(t,z))_{\mathbb{R}^{2n}} > 0 \quad \text{for} \quad |z| \geq \bar{r} \quad \text{and} \quad t \in [0,T]$$

(H_9) $\overline{\lim\limits_{|z|\to\infty}} \; |z|^{-1}|H_z(t,z)| \leq \varepsilon < \frac{1}{2}$

(H_{10}) There are constants $a_1 > 0$, $a_2 \geq 0$, and $s \in (1, \theta^{-1}]$ such that

$$H(t,z) \geq a_1|z|^s - a_2$$

for $z \in \mathbb{R}^{2n}$ and $t \in [0,T]$.

Then (4.2) possesses a T periodic solution.

Proof: Again we will obtain the result from Theorem 1.33. For convenience set $T = 2\pi$. Integrating (H_8) shows that

(4.12) $$H(t,z) \leq a_3|z|^{1/\theta} + a_4$$

for $z \in \mathbb{R}^{2n}$. Thus with E, E_1, E_2 and g as in Theorem 4.1, we have $g \in C^1(E, \mathbb{R})$ and g satisfies $(f_1)-(f_2)$. To verify (f_3), suppose $g(z_m) \leq M$ and $g'(z_m) \to 0$. Dropping subscripts as usual, for large m:

(4.13) $$M + \frac{1}{2}\|z\| \geq g(z) - \frac{1}{2}g'(z)z = \int_0^{2\pi} [H(t,z) - \frac{1}{2}(z, H_z(t,z))_{\mathbb{R}^{2n}}]\,dt$$

$$\geq (1 - \frac{1}{2\theta})\int_0^{2\pi} H(t,z)\,dt - M_1 \geq a_5\int_0^{2\pi} |z|^s\,dt - M_2 .$$

via (H_8) and (H_{10}). Define σ by $s^{-1} + \sigma^{-1} = 1$. Since $E \subset (L^\sigma(S^1))^{2n}$, the embedding being continuous, the negative norm dual $\hat{E}(= (W^{0,-1/2}(S^1))^{2n})$ of E, [15], contains $(L^s(S^1))^{2n}$ with continuous embedding. Therefore by (4.13),

(4.14) $$M_3(1 + \|z\|) \geq \|z\|_{\hat{E}}^s .$$

Since

(4.15) $$\|z\|_{\hat{E}} = \sup_{\|w\|_E \leq 1} (z, w)_{L^2} = \sup_{\|w\|_E \leq 1} [(z^0, w^0)_{L^2} + (z^-, w^-)_{L^2}]$$

it follows that

(4.16) $$\|z\|_{\hat{E}} \geq 2\pi|z^0| .$$

Therefore by (4.14) and (4.16),

(4.17) $$M_4(1 + \|z\|) \geq |z^0|^s .$$

Now (H_9) and (4.3) show that for any $\bar{\varepsilon} \in (\varepsilon, \frac{1}{2})$, there is an $M_5 = M_5(\bar{\varepsilon})$ such that

$$\|z^-\|^2 \le \|z^-\| \left(\max_{[0,2\pi]\times B_{M_5}} |H_z(t,\zeta)| + \bar{\varepsilon}\, \|z\| + 1 \right). \tag{4.18}$$

Combining (4.18) with the similar expression obtained for z^+ gives

$$\|z^+\|^2 + \|z^-\|^2 \le M_6(1 + |z^0|^2). \tag{4.19}$$

Hence by (4.14) and (4.19)

$$|z^0|^s \le M_7(1 + |z^0|) \tag{4.20}$$

which yields a bound for $|z_m^0|$ and then for (z_m) via (4.19). Thus (f_3) holds.

It remains to verify (f_6). Let $z = z^0 + z^- \in E_1$. Then as in (4.13) and (4.17),

$$g(z) \ge \|z^-\|^2 + M_8|z^0|^s - M_9 \tag{4.21}$$

which implies there is an $\alpha \in \mathbb{R}$ such that $g \ge \alpha$ on $S = E_1$. If $z \in E_2$,

$$g(z) \le \int_0^{2\pi} H(t,z)\,dt - \|z\|^2 \le a_5(\|z\|^{1/\theta} + 1) - \|z\|^2 \tag{4.22}$$

via (4.12) and (3.8). Since $\theta > \frac{1}{2}$, (4.22) shows that $g(z) \le \alpha - 1 \equiv \omega$ on $\partial Q = \partial B_R \cap E$ provided that R is chosen sufficiently large. The proof is complete.

Remark 4.23: In the subquadratic situations treated in Theorems 4.1 and 4.11, we chose $E_1 = E^0 \oplus E^-$ and $E_2 = E^+$ for their proofs. It is possible to reverse the roles of these sets and give alternate proofs obtaining possibly different critical values in the process. By doing so, however, we must work with Theorem 1.29 instead of Theorem 1.33. To illustrate we briefly sketch a second proof of Theorem 4.11. Choose E as usual, $f(z) = -g(z)$, $E_1 = E^+$, $E_2 = E^0 \oplus E^-$ with corresponding L_1, L_2 as in Theorem 3.3. The earlier proof of Theorem 4.11 shows that $f \in C^1(E,\mathbb{R})$ and satisfies $(f_1)-(f_2)$. Moreover (4.22) and (4.21) with f replacing g and inequalities reversed give (f_6) and therefore $(\bar{f}_4)$ via Lemma 1.1. Lastly f satisfies (f_5). Indeed let c be defined by (1.16) and e.g. $a = 1$. Suppose $|f(z_m) - c| \le 1$ and $f'(z_m) \to 0$ as $m \to \infty$. Then for large m,

$$|c| + 1 + \tfrac{1}{2}\|z\| \ge |f(z) - \tfrac{1}{2}f'(z)z| \ge \int_0^{2\pi} [H(t,z) - \tfrac{1}{2}(z, H_z(t,z))_{\mathbb{R}^{2n}}]\,dt \tag{4.24}$$

The boundedness of (z_m) now follows from (4.13)-(4.20).

Remark 4.25: (i) As was the case with Corollary 4.5, the conclusions of Theorem 4.11 remain valid if the inequalities involving H in (H_8) and (H_{10}) are reversed.

(ii) When H is independent of t, $(z, H_z)_{\mathbb{R}^{2n}} > 0$ on $|z| = \bar{r}$ implies that H_z has a zero in $B_{\bar{r}}$ in $\mathbb{R}^{2n}$. Thus like Theorem 4.1, Theorem 4.11 per se gives no information about the autonomous case. The following theorem, related to Theorem 4.7, addresses this point.

Theorem 4.26: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies $(H_8)-(H_{10})$. Then for all T sufficiently large, (3.2) possesses a nonconstant T periodic solution.

Proof: By (H_8), $H_z \ne 0$ if $|z| \ge \bar{r}$. Thus if ζ is a constant solution of (3.2),

$$g(\zeta) = \lambda \int_0^{2\pi} H(\zeta)\,dt \le 2\pi\lambda \max_{|z| \le \bar{r}} |H(z)| \equiv \lambda M_1 .$$

Therefore, arguing as in Theorem 4.7, it suffices to show

$$\inf_{S_v} g(z) > \lambda M_1 . \tag{4.27}$$

Greater care must exercised in choosing v since (H_7) is no longer at our disposal. For $z \in S_v$,

$$\begin{aligned} g(z) &= \lambda \int_0^{2\pi} H(z)\,dt + \|z^-\|^2 - \|v\|^2 \\ &\ge \lambda \int_0^{2\pi} a_1 |z^0 + z^- + v|^s\,dt - 2\pi a_2 + \|z^-\|^2 - \|v\|^2 \end{aligned} \tag{4.28}$$

via (H_{10}). By making appropriate choices for w in (4.15), we obtain

$$\int_0^{2\pi} |z^0 + z^- + v|^s\,dt \ge a_3[|z^0|^s + \left(\frac{\|v\|^2_{L^2}}{\|v\|}\right)^s] . \tag{4.29}$$

Setting $v = \tau u$ where $u \in \partial B_1 \cap E_2$, (4.28) - (4.29) yield

$$g(z) \ge \lambda[a_4(|z^0|^s + \tau^s) - 2\pi a_2] + \|z^-\|^2 - \tau^2 . \tag{4.30}$$

Selecting τ so large that $a_4\tau^s - 2\pi a_2 \ge 3M_1$ makes

$$g(z) \ge 3\lambda M - \tau^2 .$$

Finally taking λ so large that $\lambda M_1 \ge \tau^2$ shows $g(z) \ge 2\lambda M$ for $z \in S_v$. Hence (4.27) is satisfied and the proof is complete.

Under certain additional hypotheses, the restriction to large T in Theorem 4.26 can be removed. We present one such result which will be useful later.

<u>Corollary 4.31</u>: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies (H_8) - (H_{10}), (H_8) holding for all $z \neq 0$ and (H_{10}) with $a_2 = 0$. Then for all $T > 0$, (3.2) has a nonconstant T periodic solution.

Proof: Arguing as in Theorem 4.26, we need only choose $v = v(T)$ so that (4.27) holds with $M_1 = 0$. Under our present hypotheses (4.30) becomes

$$g(z) \ge a_4\lambda\tau^s - \tau^2 . \tag{4.32}$$

Choosing e.g. $\tau^{2-s} = \frac{1}{2}a_4\lambda$, $g(z) \ge \frac{1}{2}a_4\lambda\tau^s > 0$

Thus (4.27) is verified and the corollary is proved.

We conclude this section with the subquadratic analogues of Theorems 3.55 and 3.72. Since there are several common arguments in their proofs, we will be somewhat sketchy here.

<u>Theorem 4.33:</u> Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_4) There is a $\theta \in (\frac{1}{2}, 1)$ and $\underline{r} > 0$ such that

(H_{11}) $0 < H(z) \le \theta(z, H_z(z))_{\mathbb{R}^{2n}}$ for $0 < |z| \le \underline{r}$.

Then for all small $a > 0$, (3.2) has a $T = T(a)$ periodic solution on $H^{-1}(a)$ and $T(a) \to 0$ as $a \to 0$.

Proof: By (H_{11}), there is an $a_0 > 0$ such that for $0 < a \le a_0$, $H^{-1}(a)$ is a compact C^1 manifold which bounds a star shaped neighborhood of 0. Define $\overline{H}(z)$ as in Theorem 3.55 where τ now lies in $(1, 2)$. Then $\overline{H}$ satisfies the hypotheses of Corollary 4.31 and that result provides a nonconstant 2π periodic solution w of (3.2) with H replaced by $\overline{H}$. The proof is now essentially identical to that of Theorem 3.55 up to (3.69), the exceptions being the replacement of f by g, the search

for a lower bound for c, and the choice of $\tau \in (\theta^{-1}, 2)$. Also $a \le a_0$ here. To complete the proof we observe that (3.68) and (4.32) with $\lambda = 1$ give the lower bound for c.

Similar ideas readily establish

Theorem 4.34: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_{12}) There is a $\overline{\theta} \in (\frac{1}{2}, 1)$ and $\overline{r} > 0$ such that

$$0 < \overline{\theta}(z, H_z)_{\mathbb{R}^{2n}} \le H(z) \quad \text{for} \quad |z| \ge \overline{r}.$$

Then for all large $a > 0$, (3.2) has a $T = T(a)$ *periodic solution on* $H^{-1}(a)$ and $T(a) \to \infty$ as $a \to \infty$.

We will not carry out the details.

Remark 4.35: Under the assumptions that H is globally convex and appropriately subquadratic, Clarke and Ekeland [16] have recently obtained some existence results for (3.2) for solutions having a prescribed minimal period.

§5. A semilinear wave equation

The theory of §1 can be used to prove the existence of time periodic solutions of one dimensional semilinear wave equations. A sketch of how to do this will be given in this section. Consider the problem

$$(5.1) \qquad \begin{cases} u_{tt} - u_{xx} + g(x, u) = 0 & 0 < x < \ell \\ u(0, t) = 0 = u(\ell, t) & t \in \mathbb{R} \end{cases}$$

where we seek time periodic solutions of (5.1). For technical reasons — see [1] — we have only been able to treat the case in which the period, T, is a rational multiple of the length of the spacial interval, i.e. $T \in \ell\mathbb{Q}$. For definiteness, suppose $\ell = \pi$ and $T = 2\pi$. Letting $G(x, u)$ denote the primitive of g, formally solutions of (5.1) are critical points of

$$(5.2) \qquad \int_0^{2\pi} \int_0^{\pi} [\tfrac{1}{2}(u_t^2 - u_x^2) - G(x, u)]\, dx\, dt.$$

The indefinite nature of this functional is fairly evident. The associated action integral is

$$(5.3) \qquad A(u) = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} (u_t^2 - u_x^2)\, dx\, dt.$$

Writing

$$u = \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} a_{jk} \sin jx\, e^{ikt}$$

where $a_{j,-k} = \overline{a}_{jk}$, we find

$$A(u) = \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} (k^2 - j^2)|a_{jk}|^2.$$

Thus A is indefinite and moreover has an infinite dimensional null space in contrast to the Hamiltonian case considered earlier. This further degeneracy greatly complicates the treatment of (5.1).

Let $\mathcal{J} = [0, \pi] \times [0, 2\pi]$. Then as in §3, $L^2(\mathcal{J})$ splits into three mutually orthogonal subspaces with bases $\{\sin jx\, e^{ijt} \mid j \in \mathbb{N}\}$, $\{\sin jx\, e^{ikt} \mid j \in \mathbb{N}, k \in \mathbb{Z}, |k| > j\}$, and $\{\sin jx\, e^{ikt} \mid j \in \mathbb{N}, k \in \mathbb{Z}, |k| < j\}$. Let N^0, N^+, N^- denote respectively the subspaces of $L^2(\mathcal{J})$ spanned by these bases. Writing $u \in L^2$ as $u = u^0 + u^+ + u^- \in N^0 \oplus N^+ \oplus N^-$, we introduce a new norm on smooth such functions via

$$\|u\|^2 = A(u^+) - A(u^-) + \int_0^{2\pi}\int_0^{\pi} (u_t^0)^2\, dx\, dt .$$

The associated inner product defines a Hilbert space topology on $E = E^0 \oplus E^+ \oplus E^-$ where $E^i = \{u \in N^i \mid \|u\| < \infty\}$, $i \in \{0, +, -\}$. Thus the E^i are mutually orthogonal. Note that E^0 corresponds to the null space of A (or the wave operator). On E^0, $\|\cdot\|$ is equivalent to the $W^{1,2}$ norm while on $E^+ \oplus E^-$ (and therefore on E), $\|\cdot\| \ge \|\cdot\|_{W^{0,\frac{1}{2}}}$. The choice of norm for E^0 is somewhat arbitrary.

With E so defined, the functional given in (5.2) does not yet fit into the framework of §1, the main difficulty being the verification of (f_3). However this can be overcome by suitably modifying (5.1) and (5.2). Thus let $\beta > 0$ and for $u = u^0 + u^+ + u^- \in E$, let $v = u^0$. We replace (5.1) by

$$\left\{\begin{array}{ll} u_{tt} - u_{xx} - \beta v_{tt} + g(x,u) = 0 & 0 < x < \pi, \quad t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) & \end{array}\right. \tag{5.4}$$

and (5.2) by

$$f(u) = A(u) - \int_0^{2\pi}\int_0^{\pi} \left(\frac{\beta}{2} v_t^2 + G(x,u)\right) dx\, dt. \tag{5.5}$$

The arguments of §3 show that under suitable growth conditions on g, $f \in C^1(E, \mathbb{R})$ and satisfies (f_1)-(f_2). If g grows too rapidly, a truncation argument such as was carried out for H in §3 may also be possible — see [1]. Further hypotheses on g permit the application of one of the abstract theorems of §1. E.g. if g is strictly monotonically increasing, $g(x,0) = 0$, $g(x,y) = o(|y|)$ at $y = 0$, and $G(x,y) \le \theta y\, g(x,y)$ for large $|y|$ and some $\theta \in (0, \frac{1}{2})$, then (f_3)-(f_4) can be verified (for a problem with truncated g) as in §3 and we get a critical point of f via Theorem 0.1. It then remains, not necessarily in the following order, to let $\beta \to 0$ and to let the truncation point $\to \infty$ to get a critical point of (5.2), to show this critical point is nontrivial, and to prove it gives a classical solution of (5.1) if g is smooth. In [1], suitable estimates to carry out these steps have been obtained for certain superquadratic problems. Subquadratic cases can be treated in a similar but simpler fashion under hypotheses used in §4, however we will not carry out the details. See also [17] for some subquadratic results using different techniques.

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A variational principle of a minimax nature is developed and used to prove the existence of critical points of certain variational problems which are indefinite. The proofs are carried out directly in an infinite dimensional Hilbert space. Special cases of these problems previously had been tractable only by an elaborate finite dimensional approximation procedure. The main applications given here are to Hamiltonian systems of ordinary differential equations where the existence of time periodic solutions is established for several classes of Hamiltonians.

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